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On variations of characteristic values of entire matrix pencils[☆]

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Abstract

Entire matrix-valued functions of a complex argument (entire matrix pencils) are considered. Bounds for spectral variations of pencils are derived. In particular, approximations of entire pencils by polynomial pencils are investigated. Our results are new even for polynomial pencils. © 2005 Elsevier Inc. All rights reserved.

Keywords: Entire matrix pencils; Characteristic values; Perturbations

1. Introduction and notation

As it is well-known, matrix pencils play an essential role in various applications, see, for instance [4,11,17,15] and references therein. Perturbations of the spectrum of matrix pencils were investigated in many works, cf. [1,8–10,13], etc. Mainly, polynomial pencils were considered. In particular, the paper [1] is devoted to *linear* matrix pencils. Besides, an error bound for eigenvalues is established. In [9], stability of invariant subspaces of regular matrix pencils is considered. In [10], upper and lower *bounds* are derived for the absolute values of the eigenvalues of matrix polynomials. The bounds are based on norms of coefficient matrices. They generalize some well-known bounds for scalar polynomials and single matrices. In [13], and references given therein, perturbations of eigenvalues of diagonalizable matrix pencils *with real spectra* are investigated.

Recall that the variation of the zeros of general analytic functions under perturbations was investigated, in particular, by Rosenbloom [16]. He established conditions that provide the existence of zeros of a perturbed function in a given domain. In [5], a new approach to perturbations of scalar-valued entire functions was suggested. It is based on estimates for the norm of the resolvent of a Hilbert–Schmidt operator.

In the present paper, we consider spectrum perturbations of entire matrix pencils. Especially, we investigate approximations of entire pencils by polynomial pencils. Our results are new even in the case of polynomial pencils. They generalize the main result from [5]. It should be noted that the generalization requires additional mathematical tools. A few words about the contents. In Section 2 some auxiliary results are collected. The main result of the paper—Theorem 3.1—is presented in Section 3. The proof of Theorem 3.1 is presented in Section 4. Perturbations of polynomial matrix pencils are discussed in Section 5. In the case of polynomial pencils we improve Theorem 3.1. In Section 6, an example is given.

Let $\mathbf{C}^n$ be a Euclidean space with the Euclidean norm $\|.\|$ and the unit matrix I_n. Let A_k, B_k $(k = 1, 2, \ldots)$ be $n \times n$-matrices. Consider the matrix pencils

$$F(\lambda) = \sum_{k=0}^{\infty} \frac{A_k \lambda^k}{(k!)^\gamma} \quad (A_0 = I_n, \lambda \in \mathbf{C}) \tag{1.1a}$$

and

$$H(\lambda) = \sum_{k=0}^{\infty} \frac{B_k \lambda^k}{(k!)^\gamma} \quad (B_0 = I_n, \lambda \in \mathbf{C}) \tag{1.1b}$$

with a positive

$$\gamma \leqslant 1.$$

Assume that

$$\sum_{k=0}^{\infty} \|A_k\|^2 < \infty, \quad \sum_{k=0}^{\infty} \|B_k\|^2 < \infty. \tag{1.2}$$

Thus,

$$\theta_F := \sum_{k=1}^{\infty} A_k A_k^*$$

is an $n \times n$-matrix. The asterisk means the adjointness.

Recall that a family of matrices of the form

$$\left\{ \sum_{k=0}^{\infty} A_k \lambda^k : \lambda \in \mathbf{C} \right\},$$

where A_k; $k = 0, 1, \ldots$ are constant matrices, is called an entire pencil, if the series converges for arbitrary finite $\lambda \in \mathbf{C}$. In particular,

$$\{A_0 + \lambda A_1 : \lambda \in \mathbf{C}\}$$

is called a *linear pencil*. Put $M_F(r) := \max_{|z|=r} \|F(z)\| (r > 0)$. The limit

$$\rho(F) := \varlimsup_{r \to \infty} \frac{\ln \ln M_F(r)}{\ln r}$$

is the order of F. Relations (1.1), (1.2) mean that we consider finite order entire pencils. Indeed, if $\gamma = 1$, then we have $\rho(F) \leqslant 1$. If $\gamma < 1$, then due to Hólder inequality, from (1.1a) it follows that

$$\|F(\lambda)\| \leqslant m_0 \sum_{k=0}^{\infty} \frac{|\lambda|^k}{(k!)^\gamma} \leqslant m_0 \left[\sum_{k=0}^{\infty} 2^{-kp'} \right]^{1/p'} \left[\sum_{k=0}^{\infty} \frac{|2\lambda|^{k/\gamma}}{k!} \right]^\gamma$$
$$\leqslant m_1 e^{\gamma |2\lambda|^{1/\gamma}} \ (\gamma + 1/p' = 1),$$

where

$$m_0 = \sup_k \|A_k\|, \quad m_1 = m_0 \left[\sum_{k=0}^{\infty} 2^{-kp'} \right]^{1/p'}.$$

So function F has order no more than $1/\gamma$. We write F and H in the form (1.1), since it allows us to formulate the main result of the paper.

A zero $z_k(F)$ of det $F(z)$ is called a characteristic value of F. Our main problem is: if A_k and B_k are close, how close are the characteristic values of H to those of F? Everywhere in the present paper $\{z_k(F)\}_{k=1}^l \ (l \leqslant \infty)$ is the set of all the characteristic values of F. *If l is finite, we put $z_k^{-1}(F) = 0, k = l+1, l+2, \ldots$* . Besides, $z_k^{-1}(F)$ means $\frac{1}{z_k(F)}$.

Definition 1.1. The quantity

$$zv_F(H) = \max_j \min_k |z_k^{-1}(F) - z_j^{-1}(H)|$$

will be called the variation of characteristic values of pencil H with respect to pencil F.

Everywhere below p is a natural number satisfying the inequality

$$p > \frac{1}{2\gamma}.$$

Since $\gamma \leqslant 1$, we have $p \geqslant 1$. Furthermore, let

$$\|\theta_F\|_p := [\mathrm{Trace}(\theta_F^p)]^{1/p}$$

be the Neumann–Schatten norm of θ_F. Put

$$w_p(F) := 2\|\theta_F\|_p^{1/2} + 2[n(\zeta(2\gamma p) - 1)]^{1/2p},$$

where $\zeta(.)$ is the Riemann Zeta function. Denote also

$$\psi_p(F, y) := \sum_{k=0}^{p-1} \frac{w_p^k(F)}{y^{k+1}} \exp \left[\frac{1}{2} + \frac{w_p^{2p}(F)}{2y^{2p}} \right] \quad (y > 0) \tag{1.3}$$

and

$$q := \left[\sum_{k=1}^{\infty} \| A_k - B_k \|^2 \right]^{1/2}. \qquad (1.4)$$

2. Preliminaries

In this section, we present preliminary results which are used in the next sections.

Let A and B be linear operators in a separable Hilbert space E with a norm $\|.\|_E$. Let $\sigma(A)$ denote the spectrum of A. Then the quantity

$$sv_A(B) := \sup_{\mu \in \sigma(B)} \inf_{\lambda \in \sigma(A)} |\mu - \lambda|$$

is called the spectral variation of B with respect to A.

Denote by C_{2p} the Neumann–Schatten ideal of compact operators in E with the finite norm $\|.\|_{2p}$.

Theorem 2.1. *Let the condition*

$$A \in C_{2p} \quad (p = 1, 2, \ldots)$$

hold. Then $sv_A(B) \leqslant \tilde{y}_p(A, B)$, where $\tilde{y}_p(A, B)$ is the extreme right-hand (positive) root of the equation

$$1 = \| A - B \|_E \sum_{m=0}^{p-1} \frac{(2\|A\|_{2p})^m}{z^{m+1}} \exp\left[\frac{1}{2} + \frac{(2\|A\|_{2p})^{2p}}{2z^{2p}} \right].$$

For the proof see Theorem 8.5.4 from [6].

In particular, let $E = \mathbf{C}^n$ be the Euclidean space. The norm for matrices is understood in the sense of the operator norm. Then thanks to Theorem 4.4.1 from [6] we have:

Theorem 2.2. *Let A and B be $n \times n$-matrices. Then*

$$sv_A(B) \leqslant z(A, B),$$

where $z(A, B)$ is the extreme right-hand (unique non-negative) root of the algebraic equation

$$z^n = \| A - B \| \sum_{j=0}^{n-1} \frac{z^{n-j-1} \|A\|_2^j}{\sqrt{j!}}.$$

Below we also use the following result.

Lemma 2.3. *The extreme right-hand root z_0 of the equation*

$$z^n = P(z) := \sum_{j=0}^{n-1} c_j z^{n-j-1} \quad (c_j \equiv const \geqslant 0)$$

is non-negative and the following estimates are valid:

$$z_0 \leqslant [P(1)]^{1/n} \quad if \ P(1) \leqslant 1$$

and

$$1 \leqslant z_0 \leqslant P(1) \quad if \ P(1) \geqslant 1.$$

For the proof see Lemma 1.6.1 from [6].
In addition, we will use the following result.

Lemma 2.4. *The extreme right* (*unique positive*) *root z_a of the equation*

$$\sum_{j=0}^{p-1} \frac{1}{y^{j+1}} \exp\left[\frac{1}{2}\left(1 + \frac{1}{y^{2p}}\right)\right] = a \quad (a \equiv const > 0)$$

satisfies the inequality $z_a \leqslant \delta_p(a)$, where

$$\delta_p(a) := \begin{cases} pe/a & if \ a \leqslant pe, \\ [\ln(a/p)]^{-1/2p} & if \ a > pe. \end{cases}$$

cf. [6, Lemma 8.3.2].

3. Statement of the main result

Theorem 3.1. *Let $\psi_p(F, y)$ and q be defined by* (1.3) *and* (1.4), *respectively. Then, under conditions* (1.1), (1.2) *and $p > 1/2\gamma$ we have $zv_F(H) \leqslant r_p(F, H)$, where $r_p(F, H)$ is the unique positive* (*simple*) *root of the equation*

$$q\psi_p(F, y) = 1. \tag{3.1}$$

That is, for any characteristic value $z(H)$ of H there is a characteristic value $z(F)$ of F, such that

$$|z(H) - z(F)| \leqslant r_p(F, H)|z(H)z(F)|, \tag{3.2}$$

provided $l = \infty$. If $l < \infty$, then either (3.2) *hold or*

$$|z(H)| \geqslant \frac{1}{r_p(F, H)}. \tag{3.3}$$

The proof of this theorem is presented in the next section.

Corollary 3.2. *Let conditions* (1.1) *and* (1.2) *be fulfilled. Then* $zv_F(H) \leqslant \delta_p(F, h)$, *where*

$$\delta_p(F, H) := \begin{cases} epq & \text{if } w_p(F) \leqslant epq, \\ w_p(F)[\ln(w_p(F)/qp)]^{-1/2p} & \text{if } w_p(F) > epq. \end{cases}$$

Indeed, substitute the equality $y = xw_p(F)$ into (3.1) and apply Lemma 2.4. Then we have $r_p(F, H) \leqslant \delta_p(F, H)$. Now, the required result is due to the previous theorem.

Put

$$\Omega_j = \{z \in \mathbf{C} : q\psi_p(F, |z^{-1} - z_j^{-1}(F)|) \geqslant 1\} \quad (j = 1, \dots, l)$$

and

$$\Omega_0 = \{z \in \mathbf{C} : q\psi_p(F, 1/|z|) \geqslant 1\}$$
$$= \left\{ z \in \mathbf{C} : \sum_{k=0}^{p-1} w_p^k(F)|z|^{k+1} \exp\left[\frac{1}{2}(1 + w_p^{2p}(F)|z|^{2p})\right] \geqslant 1 \right\}.$$

Since $\psi_p(F, y)$ is monotone, Theorem 3.1 yields.

Corollary 3.3. *Under conditions* (1.1) *and* (1.2), *all the characteristic values of H are in the set* $\bigcup_{j=1}^{\infty} \Omega_j$, *provided* $l = \infty$. *If* $l < \infty$, *then all the characteristic values of H are in the set* $\bigcup_{j=0}^{l} \Omega_j$.

Let us consider approximations of an entire function H by the polynomial pencils

$$H_m(\lambda) = \sum_{k=0}^{m} \frac{B_k \lambda^k}{(k!)^\gamma} \quad (B_0 = I, \lambda \in \mathbf{C}^n; m = 1, 2, \dots).$$

Put

$$q_m(H) := \left[\sum_{k=m+1}^{\infty} \|B_k\|^2\right]^{1/2},$$

$$w_p(H_m) = 2\left\|\sum_{k=1}^{m} B_k B_k^*\right\|_p^{1/2} + 2[n(\zeta(2\gamma p) - 1)]^{1/2p}$$

and

$$\delta(p, m, H) := \begin{cases} epq_m(H) & \text{if } w_p(H_m) \leqslant epq_m(H), \\ w_p(H_m)[\ln(w_p(H_m)/pq_m(H))]^{-1/2p} & \text{if } w_p(H_m) > epq_m(H). \end{cases}$$

Define $\psi_p(H_m)$ according to (1.3). Taking, H_m instead of F in Theorem 3.1 and Corollary 3.2, we get:

Corollary 3.4. *Let H be defined by* (1.1b) *and satisfy* (1.2). *Let* $r_m(H)$ *be the unique positive root of the equation*

$$q_m(H)\psi_p(H_m, y) = 1.$$

Then either for any characteristic value $z(H)$ of H there is a characteristic value $z(H_m)$ of polynomial pencil H_m, such that

$$\left| \frac{1}{z(H)} - \frac{1}{z(H_m)} \right| \leqslant r_m(H) \leqslant \delta(p, m, H)$$

or

$$|z(H)| \geqslant \frac{1}{r_m(H)} \geqslant \frac{1}{\delta(p, m, H)}.$$

Furthermore, if $l = \infty$, relations (3.2) imply the inequalities

$$|z(F)| - |z(H)| \leqslant r_p(F, H)|z(H)||z(F)| \leqslant \delta_p(F, H)|z(H)||z(F)|.$$

Hence,

$$|z(H)| \geqslant (r_p(F, H)|z(F)| + 1)^{-1}|z(F)| \geqslant (\delta_p(F, H)|z(F)| + 1)^{-1}|z(F)|.$$

This inequality yields the following result.

Corollary 3.5. *Under conditions (1.1), (1.2) and $l = \infty$, for a positive number R_0, let F have no characteristic values in the disc $\{z \in \mathbf{C} : |z| \leqslant R_0\}$. Then H has no characteristic values in the disc $\{z \in \mathbf{C} : |z| \leqslant R_1\}$ with*

$$R_1 = \frac{R_0}{\delta_p(F, H)R_0 + 1} \quad or \quad R_1 = \frac{R_0}{r_p(F, H)R_0 + 1}.$$

Let us assume that under (1.1), there is a constant $d_0 \in (0, 1)$, such that

$$\overline{\lim_{k \to \infty}} \sqrt[k]{\|A_k\|} < 1/d_0 \quad \text{and} \quad \overline{\lim_{k \to \infty}} \sqrt[k]{\|B_k\|} < 1/d_0$$

and consider the functions

$$\tilde{F}(\lambda) = \sum_{k=0}^{\infty} \frac{A_k (d_0 \lambda)^k}{(k!)^\gamma} \quad \text{and} \quad \tilde{H}(\lambda) = \sum_{k=0}^{\infty} \frac{B_k (d_0 \lambda)^k}{(k!)^\gamma}.$$

That is, $\tilde{F}(\lambda) \equiv F(d_0 \lambda)$ and $\tilde{H}(\lambda) \equiv H(d_0 \lambda)$. So functions $\tilde{F}(\lambda)$ and $\tilde{H}(\lambda)$ satisfy conditions (1.2). Moreover,

$$w_p(\tilde{f}) = 2 \left\| \sum_{k=1}^{\infty} d_0^{2k} A_k (A^k)^* \right\|_p^{1/2} + 2[n(\zeta(2\gamma p) - 1)]^{1/2p}.$$

Since

$$\sum_{k=1}^{\infty} d_0^{2k} \|A_k - B_k\|^2 < \infty,$$

we can directly apply Theorem 3.1 and Corollary 3.2 taking into account that $d_0 z_k(\tilde{f}) = z_k(F)$, $d_0 z_k(\tilde{H}) = z_k(H)$.

4. Proof of Theorem 3.1

For a finite integer m, consider the matrix polynomials

$$P(\lambda) = \sum_{k=0}^{m} \frac{A_k \lambda^{m-k}}{(k!)^{\gamma}} \quad \text{and} \quad Q(\lambda) = \sum_{k=0}^{m} \frac{B_k \lambda^{m-k}}{(k!)^{\gamma}} \quad (A_0 = B_0 = I). \tag{4.1}$$

In addition, $\{z_k(P)\}_{k=1}^n$ and $\{z_k(Q)\}_{k=1}^m$ are the sets of all the characteristic values of P and Q, respectively, taken with their multiplicities. Introduce the block matrices

$$\tilde{A}_m = \begin{pmatrix} -A_1 & -A_2 & \dots & -A_{m-1} & -A_m \\ \frac{1}{2^{\gamma}} I_n & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{3^{\gamma}} I_n & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & \frac{1}{m^{\gamma}} I_n & 0 \end{pmatrix}$$

and

$$\tilde{B}_m = \begin{pmatrix} -B_1 & -B_2 & \dots & -B_{m-1} & -B_m \\ \frac{1}{2^{\gamma}} I_n & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{3^{\gamma}} I_n & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & \frac{1}{m^{\gamma}} I_n & 0 \end{pmatrix}.$$

Lemma 4.1. *The relation* $\det P(\lambda) = \det(\lambda I_{mn} - \tilde{A}_m)$ *is true.*

Proof. Let z_0 be a characteristic value of P. Then

$$\sum_{k=0}^{m} \frac{z_0^{m-k}}{(k!)^{\gamma}} A_k v = 0,$$

where v is the corresponding eigenvector of P. Put

$$x_k = \frac{z_0^{m-k}}{(k!)^{\gamma}} v \quad (k = 1, \dots, m).$$

Then

$$z_0 x_k = x_{k-1}/k^{\gamma} \quad (k = 2, \dots, m)$$

and

$$\sum_{k=0}^{m} \frac{z_0^{m-k}}{(k!)^{\gamma}} A_k v = \sum_{k=1}^{m} A_k x_k + z_0 x_1 = 0.$$

So vector $x = (x_1, \ldots, x_m)$ satisfies the equation $\tilde{A}_m x = z_0 x$. If the spectrum of $P(.)$ is simple, the lemma is proved. If $\det P(.)$ has non-simple roots, then the required result can be proved by a small perturbation. $\square$

Put

$$q(P, Q) := \left[\sum_{k=1}^{m} \| A_k - B_k \|^2 \right]^{1/2}$$

and

$$w_p(P) := 2 \left\| \sum_{k=1}^{m} A_k A_k^* \right\|_p^{1/2} + 2[n(\zeta(2\gamma p) - 1)]^{1/2p}$$

for a natural $p > 1/2\gamma$; $\psi_p(P, y)$ is defined according to (1.3).

Lemma 4.2. *For any characteristic value $z_m(Q)$ of $Q(z)$, there is a characteristic value $z(P)$ of P, such that*

$$|z(P) - z(Q)| \leqslant r_p(Q, P),$$

where $r_p(Q, P)$ be the unique positive root of the equation

$$q(P, Q)\psi_p(P, y) = 1. \tag{4.2}$$

Proof. Due to the previous lemma

$$\lambda_k(\tilde{A}_m) = z_k(P), \quad \lambda_k(\tilde{B}_m) = z_k(Q) \quad (k = 1, 2, \ldots, mn), \tag{4.3}$$

where $\lambda_k(\tilde{A}_m)$, $\lambda_k(\tilde{B}_m)$, $k = 1, \ldots, nm$ are the eigenvalues with their multiplicities of $\tilde{A}_m$ and $\tilde{B}_m$, respectively. Clearly,

$$\| \tilde{A}_m - \tilde{B}_m \| = q(P, Q).$$

Due to Theorem 2.1, for any $\lambda_j(\tilde{B}_m)$, there is a $\lambda_i(\tilde{A}_m)$, such that

$$|\lambda_j(\tilde{B}_m) - \lambda_i(\tilde{A}_m)| \leqslant y_p(\tilde{A}_m, \tilde{B}_m), \tag{4.4}$$

where $y_p(\tilde{A}_m, \tilde{B}_m)$ is the unique positive root of the equation

$$\| \tilde{A}_m - \tilde{B}_m \| \sum_{k=0}^{p-1} \frac{(2\|\tilde{A}_m\|_{2p})^k}{y^{k+1}} \exp\left[\left(1 + \frac{(2\|\tilde{A}_m\|_{2p})^{2p}}{y^{2p}} \right) \Big/ 2 \right] = 1.$$

But $\tilde{A}_m = M + C$, where

$$M = \begin{pmatrix} -A_1 & -A_2 & \dots & -a_{m-1} & -a_m \\ 0 & 0 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \frac{1}{2^\gamma} I_n & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{3^\gamma} I_n & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & \frac{1}{m^\gamma} I_n & 0 \end{pmatrix}.$$

Therefore, with the notation

$$c = \sum_{k=1}^{m} A_k A_k^*,$$

we have

$$MM^* = \begin{pmatrix} c & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \text{and} \quad CC^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & I_n/2^{2\gamma} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 0 & I_n/m^{2\gamma} \end{pmatrix}.$$

Clearly,

$$\|M\|_{2p}^2 = \left\| \sum_{k=1}^{m} A_k A_k^* \right\|_p.$$

In addition,

$$\text{Trace}(CC^*)^p = n \sum_{k=2}^{m} 1/k^{2p\gamma}.$$

Thus,

$$\|C\|_{2p} = \left[n \sum_{k=2}^{m} \frac{1}{k^{2p\gamma}} \right]^{1/2p}.$$

Hence

$$\|\tilde{A}_m\|_{2p} \leqslant \left\| \sum_{k=1}^{m} A_k \, A_k^* \right\|_p^{1/2} + \left[n \sum_{k=2}^{m} \frac{1}{k^{2p\gamma}} \right]^{1/2p}. \tag{4.5}$$

This and (4.3) prove the lemma. □

Proof of Theorem 3.1. Consider the polynomial pencils

$$F_m(\lambda) = \sum_{k=0}^{m} \frac{A_k \lambda^k}{(k!)^\gamma} \quad \text{and} \quad H_m(\lambda) = \sum_{k=0}^{m} \frac{B_k \lambda^k}{(k!)^\gamma}. \tag{4.6}$$

Clearly, $\lambda^m F_m(1/\lambda) = P(\lambda)$ and $H_m(1/\lambda)\lambda^m = Q(\lambda)$. So

$$z_k(P) = 1/z_k(F_m), \quad z_k(Q) = 1/z_k(H_m). \tag{4.7}$$

Take into account that the roots continuously depend on coefficients, we have the required result, letting in the previous lemma $m \to \infty$. $\quad\square$

5. Perturbations of polynomial pencils

In this section, we improve Theorem 3.1 in the case of polynomial pencils. Again consider the polynomial pencils defined by (4.1). Put

$$\eta(P) = \left[\text{Trace} \left(\sum_{k=1}^{m} A_k A_k^* \right) + n(m-1) \right]^{1/2}$$

and

$$\psi(P, y) := \sum_{k=0}^{nm-1} \frac{\eta^k(P)}{y^{k+1}\sqrt{k!}} \quad (y > 0).$$

In addition, as above

$$q(P, Q) = \left[\sum_{k=1}^{m} \|A_k - B_k\|^2 \right]^{1/2}.$$

Theorem 5.1. *Let P and Q be defined by* (4.1). *Then for any characteristic value $z(Q)$ of $Q(z)$, there is a characteristic value $z(P)$ of $P(z)$, such that*

$$|z(P) - z(Q)| \leqslant r(P, Q), \tag{5.1}$$

where $r(P, Q)$ is the unique positive root of the equation

$$q(P, Q)\psi(P, y) = 1. \tag{5.2}$$

Proof. Take matrices $\tilde{A}_m$, $\tilde{B}_m$, defined in Section 3, with $\gamma = 0$. Due to Theorem 2.2, for any $\lambda_j(\tilde{B}_m)$, there is a $\lambda_i(\tilde{A}_m)$, such that

$$|\lambda_j(\tilde{B}_m) - \lambda_i(\tilde{A}_m)| \leqslant x(\tilde{A}_m, \tilde{B}_m),$$

where $x(\tilde{A}_m, \tilde{B}_m)$ is the unique positive root of the equation

$$\|\tilde{A}_m - \tilde{B}_m\| \sum_{k=0}^{mn-1} \frac{\|\tilde{A}_m\|_2^k}{y^{k+1}\sqrt{k!}} = 1.$$

Clearly, $\|\tilde{A}_m\|_2 = \eta(P)$. Hence, $x(\tilde{A}_m, \tilde{B}_m) \leqslant r(P, Q)$. This and (4.3) proves the theorem. $\quad\square$

Denote

$$p_0 := q(P, Q) \sum_{k=0}^{mn-1} \frac{\eta^k(P)}{\sqrt{k!}} = q(P, Q)\psi(P, 1)$$

and

$$\delta_0(P, Q) := \begin{cases} \sqrt[mn-1]{p_0} & \text{if } p_0 \leqslant 1, \\ p_0 & \text{if } p_0 > 1. \end{cases}$$

Due to Lemma 2.3, $r(P, Q) \leqslant \delta_0(P, Q)$. Now Theorem 5.1 implies:

Corollary 5.2. *Let P and Q be defined by* (4.1). *Then for any characteristic value $z(Q)$ of $Q(z)$, there is a characteristic value $z(P)$ of $P(z)$, such that $|z(P) - z(Q)| \leqslant \delta_0(P, Q)$.*

In particular, let

$$P(\lambda) = \lambda I + A_1, \quad Q(\lambda) = \lambda I + B_1. \tag{5.3}$$

Then $\eta^2(P) = \text{Trace}(A_1 A_1^*) = \|A_1\|_2^2$ and

$$\psi(P, y) = \psi_1(y) := \sum_{k=0}^{n-1} \frac{\|A_1\|_2^k}{y^{k+1}\sqrt{k!}} \quad (y > 0).$$

In addition, $q(P, Q) = q_1 := \|A_1 - B_1\|$. Due to Theorem 5.1, for any $z(Q)$ there is a $z(P)$, such that $|z(P) - z(Q)| \leqslant r_0$, where r_0 is the unique positive root of Eq. (5.2) with $q(P, Q) = q_1$ and $\psi(P, y) = \psi_1(y)$. Denote $p_1 := q_1\psi_1(1)$ and

$$\delta_1 := \begin{cases} \sqrt[n-1]{p_1} & \text{if } p_1 \leqslant 1, \\ p_1 & \text{if } p_1 > 1. \end{cases}$$

Due to Corollary 5.2, under (5.3), for any $z(Q)$, there is a $z(P)$ such that $|z(P) - z(Q)| \leqslant \delta_1$.

Let us derive bounds for the characteristic values of Q. Let $B_j = (b_{sk}^{(j)})_{s,k=1}^n$ $(j = 1, \ldots, m)$ be arbitrary $n \times n$-matrices.

In addition, let v_j, w_j and d_j be the *upper nilpotent, lower nilpotent and diagonal parts* of B_j. So $B_j = v_j + d_j + w_j$. Take $A_j = v_j + d_j$. That is, A_j is the upper triangular part of B_j and P is the upper triangular part of Q. Moreover, we have

$$q(P, Q) \leqslant q_w := \left[\sum_{k=1}^m \|w_k\|^2\right]^{1/2},$$

$$\eta(P) = \eta_d := \left[\sum_{j=1}^m \sum_{1 \leqslant l \leqslant k \leqslant n} |b_{lk}^{(j)}|^2 + n(m-1)\right]^{1/2},$$

$$p_0 = \tilde{p} := q_w \sum_{k=0}^{mn-1} \frac{\eta_d^k}{\sqrt{k!}}.$$

and $\delta_0(P, Q) = \tilde{\delta}$, where

$$\tilde{\delta} := \begin{cases} \sqrt[mn-1]{\tilde{p}} & \text{if } \tilde{p} \leqslant 1, \\ \tilde{p} & \text{if } \tilde{p} > 1. \end{cases} \tag{5.4}$$

Since P is a triangular matrix, characteristic values of P are the roots $R_{1k}, \ldots, R_{mk}$ of the diagonal polynomials

$$\lambda^m + b_{kk}^{(1)} \lambda^{m-1} + \cdots + b_{kk}^{(m)} \quad (k = 1, \ldots, n).$$

Due to Corollary 5.2, we get:

Corollary 5.3. *All the characteristic values of Q lie in the union of the sets*

$$\{z \in \mathbf{C} : |z - R_{jk}| \leqslant \tilde{\delta}\} \quad (j = 1, \ldots, m; \ k = 1, \ldots, n).$$

Let us compare this result with the well-known generalized Hadamard theorem, cf. [2,3] and [4, Section 14.5]. As it was mentioned above, the spectra of P and block matrix $\tilde{A}_m$ with $\gamma = 0$ coincide. The generalized Hadamard theorem does not assert that the block matrix $\tilde{A}_m$ with non-singular triangular blocks A_j is invertible. That is, it does not assert that the pencil P is invertible, if z is not a root of the diagonal entries of P. At the same time Corollary 5.3 asserts that the pencil P is invertible, provided z is not a root of the diagonal entries of P. Thus, our results improve the generalized Hadamard theorem when coefficients of pencils are "close" to triangular matrices.

6. Example

Let us consider the pencil

$$H(z) = I_n + C_1 z + z^2 e^{zv} C_2 \quad (0 < v = const < 1)$$

with $n \times n$-matrices C_1, C_2. As it is well-known, such matrix quasipolynomials play an essential role in the theory of differential-difference equations, cf. [11]. Rewrite this function in the form (1.1b) with $\gamma = 1$, and

$$B_1 = C_1, \quad B_k = C_2 v^{k-2} k(k-1) \quad (k = 2, 3, \ldots).$$

Put $H_2(\lambda) = I_n + C_1 z + C_2 z^2$ and

$$q_2(H) = \|C_2\| \left[\sum_{k=3}^{\infty} v^{2(k-2)} k^2 (k-1)^2 \right]^{1/2}.$$

This series is easily calculated. Furthermore, put

$$w_1(H_2) = 2[\text{Trace}(C_1 C_1^* + 4C_2 C_2^*)]^{1/2} + 2[n(\zeta(2) - 1)]^{1/2}.$$

Then due to Corollary 3.3, we can assert that all the zeros of H are in the set $\bigcup_{j=0}^{2n} \Omega_j$, where

$$\Omega_0 = \{z \in \mathbf{C} : q_2(H)|z| \exp[(1 + w_1^2(H_2)|z|^2)/2] \geqslant 1\}$$

and

$$\Omega_j = \left\{ z \in \mathbf{C} : q_2(H) e^{1/2} |z_j^{-1}(H_2) - z^{-1}|^{-1} \exp\left[\frac{w_1^2(H_2)}{2|z_j^{-1}(H_2) - z^{-1}|^2} \right] \geqslant 1 \right\}$$
$$(j = 1, \ldots, 2n).$$

Besides, $z_j(H_2)$ are the roots of the polynomial $\det H_2(\lambda) = \det (I_n + C_1 z + C_2 z^2)$.

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